TOPICS IN COMPLEX ANALYSIS @ EPFL, FALL 2024
HOMEWORK 8

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Homework 8.1 (On Picard's little theorem)**.**

a. Employ Proposition 5.10 from the lecture notes to prove Picard's little theorem without using Picard's great theorem.
b. Show Picard's little theorem is equivalent to the following claim. If $f, g\colon \mathbf{C} \to \mathbf{C}$ are entire functions such that $e^f + e^g = 1$, then f, g are constant.

Homework 8.2 (Landau's sharpened version of Picard's little theorem*)**.**

a. Prove there exists a function $R\colon \mathbf{C} \setminus \{0, 1\} \to (0, \infty)$[1] such that
$$\{f \in \mathcal{H}(\bar{B}_{R(a)}(0)) : f(0) = a,\ f'(0) = 1,\ f \text{ omits 0 and 1}\} = \emptyset.$$
b. Show that the statement in a. implies Picard's little theorem.

Homework 8.3 (Approaching Picard's great theorem)**.** Prove Picard's great theorem is equivalent to the following claim. Let $f\colon B_1(0) \setminus \{0\} \to \mathbf{C}$ be any holomorphic function such that $\{0, 1\} \cap f(B_1(0) \setminus \{0\}) = \emptyset$. Then f or $1/f$ is bounded in a neighborhood of 0.

Homework 8.4 (A stronger version of the sharpened Montel theorem)**.** Let $G \subset \mathbf{C}$ be a simply connected domain and for $m \in \mathbf{N}$ define
$$\mathcal{F}_m := \{f\colon G \to \mathbf{C} : f \text{ holomorphic},\ f(G) \cap \{0\} = \emptyset,\ \#\{f = 1\} \leq m\}.$$
Let $\{f_n\}_{n\in\mathbf{N}}$ be a sequence in $\mathcal{F}_m$. Show that either along the entire sequence $|f_n|$ converges locally uniformly to ∞ or there exists a subsequence of $(f_n)_{n\in\mathbf{N}}$ that converges locally uniformly to a holomorphic function $f\colon G \to \mathbf{C}$[2].

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[1] **Hint.** Set $R(a) := 3L(1/2, |a|)$, where L is given by Schottky's theorem.

[2] **Hint.** Consider a suitable root $\sqrt[k]{f}$ for $f \in \mathcal{F}_m$.